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On Gravitational Shock Waves

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Abstract

The discontinuity planes of the Riemann curvature tensor R^i_{klm} in the Einsteinian vacuum $R_{kl} = 0$ are isotropic hypersurfaces. These surfaces are to be conceived as being constructed of lightlike geodesics, which form, in the eikonal approximation, gravitational radiation. The discontinuity planes themselves describe the wave fronts of disturbances of the metric g_{ik}, propagating with the velocity of light. By successively applying continuity conditions for the derivatives of the g_{ik} that follow from Einstein's equations, we obtain the universal expression of gravitational wave fields in space-time "strips" (or representations) of arbitrarily selected Einstein spaces.

On the ground of general covariance, the ten Einstein field equations leave four functions $\varphi^i(y)$, undetermined; hence only six of these equations are independent. The symmetric Ricci tensor, however, as well as the metric, have each ten algebraically independent components. Therefore, there must exist four differential identities between the R_{ik}, which, invoking Bianchi, assume the form

$$S^k_{i,k} + S_i{}^m \Gamma^l_{ml} - S_k{}^l \Gamma^k_{il} = 0 \tag{1}$$

by using preferably Einstein's tensor $S_{ik}(i, k = 0, 1, 2, 3)$. Our problem is to construct a Cauchy problem whereby, e.g., at time $y^0 = ct = 0$ values will be prescribed. To render this manipulation tractable, let us separate in Bianchi's conditions the derivatives with respect to time $y^0 = ct$ and sum up the remaining terms on the right-hand side abridged B_k:

$$S^0_{k,0} = -S^\rho_{k,\rho} + S_m{}^n \Gamma^m_{k,n} - S_k{}^n \Gamma^l_{nl} \equiv B_k \tag{2}$$

The equations (2) are identities where B_k does not contain derivatives with respect to time higher than to the second order; the same must hold for the left-hand side; hence

$$S_k{}^0 \equiv R_k{}^0 - \tfrac{1}{2}\delta_k{}^0 R \tag{3}$$

does not contain second-order time derivatives. A further examination would show that no second-order time derivatives of g_0 and g_{00} are to be found in the whole relation. Thus, $S_k{}^0$ contains no time-derivative of g_{k0} whatsoever.

Furthermore, Einstein's field equations are in involution,[1] meaning that if the six space-space components or the Ricci tensor vanish everywhere in a domain

$$R_{\mu\nu} = 0 \qquad (\mu, \nu = 1, 2, 3)$$

as well as the four right-hand sides of Ricci's identities at time $t = 0$, then Einstein's field equations are verified in the whole domain. This implies that if $S_k{}^0 = 0$ at some fixed time, it remains so at all times.

We can therefore use the equation $S_k{}^0 = 0$ in order to determine algebraically g_{k0}, and then eliminate g_{k0} from the six remaining field equations, which will then contain only the six functions $g_{\mu\nu}$ $(\mu, \nu = 1, 2, 3)$ besides first and second derivatives. And since general covariance allows to choose arbitrarily four functions, let us select them in such a way as to get rid of four suitable combinations of the $g_{\mu\nu}$: In short, $S_k{}^0 = 0$, eliminate the four g_{k0} and by an appropriate choice of the coordinates, eliminate four combinations of the $g_{\mu\nu}$; only *two essential field functions* remain. If they are determined by Einstein's field equations, then the whole field is determined, too.

The main argument will now be to construct these two essential functions (cf. Bondi, 1966).

To illustrate this, let D be a domain of space-time V_4 and let a hypersurface Σ in D be defined by the equation

$$\Sigma_{\text{Def.}}: Z = 0$$

separating D into two partial domains:

$$D_+, \qquad \text{where } Z > 0,$$

$$D_-, \qquad \text{where } Z < 0$$

We shall assume for the sake of simplicity that the $g_{ik}(y)$ are everywhere holomorphic functions in D except on Σ, where the transversal derivatives of a certain order n and higher might be discontinuous (Treder, 1962):

$$\left[\frac{\partial^m}{\partial Z^m} g_{ik}\right] \equiv \left(\frac{\partial^m}{\partial Z^m} g_{ik}\right)_{Z \to +0} - \left(\frac{\partial^m}{\partial Z^m} g_{ik}\right)_{Z \to -0} \neq 0 \quad \text{for } m \geqslant n \tag{4}$$

Nevertheless, the field equations $R_{ik} = 0$ for the vacuum are supposed to apply everywhere. Such discontinuities of the derivatives of the metric must exist, if

[1] Involution in the sense in which Cartan used the term.

the gravitational field is to be modulated so as to propagate information; Σ is thus a wave front.

We shall now show that such new functions really do not exist unless Σ is an isotropic hypersurface. In order to see this, let us, however, develop g_{ik} by adapting the development to the situation, i.e., to the existence of the wave front:

$$g_{ik} = g^-_{ik} + \underset{(n)}{\gamma_{ik}} \underset{(n)}{H}(Z) + \underset{(n+1)}{\gamma_{ik}} \underset{(n+1)}{H}(Z) + \cdots \qquad (5)$$

Here, g^-_{ik} designates the metric in the partial domain D_-, where $Z < 0$, as well as its analytical continuation in the domain D_+; the functions $\underset{(m)}{\gamma_{ik}}$ may very well be holomorphic, whereas $\underset{(m)}{H}(Z)$ denotes discontinuous functions by passing through Σ(Heaviside function)

$$\left.\begin{aligned} \underset{(m)}{H}(Z) &= \frac{Z^m}{m!} \text{ for } Z \geqslant 0 \\ &= 0 \text{ for } Z < 0 \end{aligned}\right\} \text{ for all } m \geqslant n$$

In the lowest order of the discontinuity, we obtain

$$(\underset{(n)}{\gamma_{ik}})_{Z=0} = \left[\frac{\partial^n}{\partial Z^n} g_{ik}\right]$$

A corresponding development of the Ricci tensor R_{ik} will indeed exist. Since Einstein's field equations are valid throughout D, the coefficients of this latter sequence must all vanish, in order to warrant $R_{ik} = 0$:

$$R_{ik} = \frac{\partial}{\partial Z} R_{ik} = \cdots = \frac{\partial^m}{\partial Z^m} R_{ik} = \cdots = 0 \qquad (6)$$

whence

$$[R_{ik}] = \left[\frac{\partial}{\partial Z} R_{ik}\right] = \left[\frac{\partial^m}{\partial Z^m} R_{ik}\right] = \cdots = 0 \qquad (7)$$

The first of the latter equations not to be identically satisfied is – by reason of R_{ik} containing second derivatives of the metric –

$$\left[\frac{\partial^{n-2}}{\partial Z^{n-2}} R_{ik}\right] = 0 \qquad (8)$$

According to (8) the $\underset{(n)}{\gamma_{ik}}$ are unequivocally conditioned or specified.

Now, if Σ is either timelike or spacelike—meaning that the normal vector p_i satisfies the inequality

$$g^{ik} p_i p_k \neq 0$$

then equation (8) contains the following six algebraically independent relations:

$$p^l p_l \underset{(n)}{\gamma_{ik}} + g^{ml} \underset{(n)}{\gamma_{ml}} p_i p_k - \underset{(n)}{\gamma_{kl}} p^l p_i = 0 \tag{9}$$

The general solution $\underset{(n)}{\gamma_{ik}}$ depends consequently upon four functions and can be put as

$$\underset{(n)}{\gamma_{ik}} = \underset{(n)}{a_i} p_k + \underset{(n)}{a_k} p_i \tag{10}$$

But then the first discontinuity in the development of the metric can be eliminated by changing the coordinates to

$$\bar{y}^i = y^i + \underset{(n+1)}{a^i} \; H(Z) + \cdots \tag{11}$$

and the next discontinuity takes the place of the eliminated one. However, the same reasoning applies to the new one, and so forth. Hence, discontinuities of that sort are only due to fortuitous choices of coordinates and can be got rid of by changing them. Such fortuitous discontinuities are not essential, they are mere appearances which can be removed and cannot, of course, furnish any real information.

The case of lightlike (isotropic) hypersurfaces is quite different, i.e., if

$$g^{ik} p_i p_k = 0$$

for then equation (8) yields four linear algebraic conditions for the $\underset{(n)}{\gamma_{ik}}$ on Σ:

$$-\underset{(n)}{\gamma_{ik}} p^k + \tfrac{1}{2} \underset{(n)}{\gamma_{kl}} g^{kl} p_i = 0 \tag{12}$$

From equation (12) it follows that the $\underset{(n)}{\gamma_{ik}}$ are found to consist (i) of a term which again can be transformed through change of coordinates to the elimination of a fortuitous discontinuity, as well as (ii) of a second part which depends upon two functions, say $\underset{(n)}{E}$ and $\underset{(n)}{\delta}$. Hence, the discontinuity is found to be perpendicular to the propagation vector p_i,

$$\underset{(n)}{\gamma_{ik}} = A_i p_k + A_k p_i + \pi_i \pi_k + \tilde{\pi}_i \pi_k \tag{13}$$

where

$$\pi_k p^k = \tilde{\pi}_k p^k = \pi_k \tilde{\pi}^k = p^k p_k = 0$$

the π_k are given by *Stellmacher's Ansatz* (Stellmacher, 1938):

$$\underset{(n)}{\pi_k} = (\tfrac{1}{2})^{1/4} \underset{(n)}{E}{}^{1/4} (e^{i\underset{(n)}{\delta}} \omega_k^*) \tag{14a}$$

$$\underset{(n)}{\tilde{\pi}_k} = (\tfrac{1}{2})^{1/4} E^{1/4} (e^{i\underset{(n)}{\delta}} \omega_k - e^{-i\underset{(n)}{\delta}} \omega_k^*) \tag{14b}$$

Therefore, on isotropic hypersurfaces, essential discontinuities can indeed exist, depending on two further parameters. These discontinuities are the so-called *shock waves*!

Although the relations (8) are, at first sight, conditions only on Σ, nevertheless the $\underset{(n)}{\gamma}_{ik}$ can be required to fulfill all conditions in the whole space G, implying the relations

$$\left[\frac{\partial^{n-1}}{\partial Z^{n-1}} R_{ik}\right] = 0 \tag{15}$$

Four of these ten equations are then identically satisfied, whereas two equations are homogeneous linear differential equations for the quantities $\underset{(n)}{E}$ and $\underset{(n)}{\delta}$. Their propagation along the null geodesics that generate the hypersurface Σ is determined by

$$\frac{\partial}{\partial x^l}(\sqrt{-g^{1/2}}\underset{(n)}{E}p^l) = 0 \tag{16}$$

$$\underset{(n)}{\delta}{}_{,l}p^l = \frac{d}{dt}\underset{(n)}{\delta} = 0 \tag{17}$$

where

$$p^l p_l = 0, \qquad p^i{}_{;l}p^l = 0$$

The four remaining equations yield now the structure of inhomogeneous algebraic equations allowing us to determine the $\underset{(n+1)}{\gamma}_{ik}$. Therefore, $\underset{(n+1)}{\gamma}_{ik}$ reveal a homogeneous part of the same algebraic structure as the $\underset{(n)}{\gamma}_{ik}$. Hence the $\underset{(n+1)}{\gamma}_{ik}$ depend, in the same sense, on two functions $\underset{(n+1)}{E}$ and $\underset{(n+1)}{\delta}$. With the requirement that these equations on Σ are valid for the whole G, we are able to continue this procedure, getting in every case the functions $\underset{(m)}{E}$ and $\underset{(m)}{\delta}$ with the corresponding linear differential equations, which determine explicitly the propagation along the characteristic rays. [The result obtained for the order $n + 1$ is independent of the choice of $\underset{(m)}{\gamma}_{ik}$ ($m > n + 1$) in G (Treder, 1962, Sections 6 and 7).]

The field $g^{+}_{ik} - g^{-}_{ik}$ which is of the character of a gravitational disturbance (shock waves) in G depends on $\underset{(m)}{E}$ and $\underset{(m)}{\delta}$, each of these sets forming a so-called "news-function"; their initial data can be given, in an arbitrary way, on the two-dimensional section of the isotropic hypersurface Σ with a spacelike cone x^0 = const. On the total Σ these functions are defined by the propagation equation, and, corresponding to the isotropy of Σ, the disturbances are propagated in V_3 with the local velocity of light. The discontinuity of lowest order is interpreted in a geometrico-optical way, $\underset{(n)}{E}$ being the intensity and $\underset{(n)}{\delta}$ the polarization angle of the propagated disturbance, which shows some similarity with the laws of geometrical optics.

The above analysis proves that the Einsteinian equations admit of a modulation in the gravitational field with information propagating with the velocity of light corresponding to the isotropy of the wave front Σ. The waves are purely transversal – a fact that determines them completely by giving an amplitude and a polarization angle as well as the wave front and the direction of propagation. All these quantities can be furnished in terms of a lightlike coordinate Z, since the choices of $\underset{(m)}{E}$ and $\underset{(m)}{\delta}$ are independently made.

A Taylor expansion [equation (5)] of g_{ik} induces an analogous series for the Riemannian tensor

$$R_{iklm} = R^{-}_{iklm} + \underset{(n-2)}{P_{iklm}} H(Z) + \cdots$$

R^{-}_{iklm} is defined as the Riemannian tensor for $Z < 0$ or is conceived of as being an analytical continuation into the domain $Z > 0$. The coefficients P_{iklm} are holomorphic functions, the lowest one satisfying the algebraic relations

$$\underset{(n-2)}{P_{iklm}}\, p^m = 0 \tag{18}$$

$$\epsilon^{iklm} \underset{(n-2)}{P_{stlm}}\, p_i \tag{19}$$

where ϵ is Levi-Civita's symbol. Conversely, these equations render it possible to define the existence of a gravitational wave front Σ.

Let us finally consider the propagation of a gravitational wave in the flat Minkowski space E_4. In the simplest case, a wave front exists on a Minkowskian null hypersurface $Z = 0$, the space being flat on the one side ($R^{-}_{iklm} = 0$) with a nonvanishing R_{iklm} on the other side. In the neighborhood of the wave front as well as on the wave front itself obtains

$$R_{iklm} p^m = 0 \tag{20}$$

$$\epsilon^{iklm} P_{stlm} p_i \tag{21}$$

where

$$p^i p_i = g^{ik} p_i p_k = 0$$

If these conditions are fulfilled in the whole wave domain, one speaks of gravitational waves with "flat" fronts (Brinkmann, 1923); these waves are indeed "flat gravitational waves," if and only if a coordinate system can be chosen such as to make the g_{ik} dependent solely upon the isotropic coordinate Z. With $Z \equiv x^0$, we get

$$(g_{ik}) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & g_{11}(x^0) & g_{12}(x^0) & 0 \\ 0 & g_{12}(x^0) & g_{22}(x^0) & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{22a}$$

for which a field equation between g_{11}, g_{12}, and g_{22} reads

$$g^{ik}g_{ik,00} + \tfrac{1}{2}g^{ik}{}_{,0}g_{ik,0} = 0 \tag{22b}$$

The very question of the real existence (and of experimental evidence) of gravitational radiation is still an open issue. Some physicists are strongly convinced of its very existence. However, the crucial experiments of the "receptions of gravitation waves from the universe" are very controversial. Furthermore, the theoretical arguments for the possibility of an emission of gravitational waves by means of freely moving bodies in a gravitational field are rather bizarre. Free motions in gravitation fields are forceless motions, which we can describe by acceleration-free rest systems in Fermi coordinates. Free gravitational waves may therefore—according to Infeld—exist from "the first moments of the universe," but there cannot exist *sources* of gravitational radiations.

In connection with the exact plane gravitational waves Einstein and Rosen (1936) have proved that the form $g_{ik} = g_{ik}(z)$ [see equations (22a) (22b)] is incompatible with the de Donder condition $(\partial/\partial x^k)\sqrt{-g^{1/2}}g^{ik} = 0$ according to the exact equations $R_{ik} = 0$. But the de Donder condition excludes the nonphysical modes of the plane waves in the linearized theory (Einstein, 1916). And thus, cogently reasoning, Einstein writes to Max Born: "Ich habe zusammen mit einem jungen Mitarbeiter gefunden, dass es keine Gravitationswellen gibt, trotzdem man dies gemäß der ersten Approximation für sicher hielt" (Einstein, 1969).[1]

References

Bondi, H. (1966). In *Einstein-Symposium*, p. 119, Akademie-Verlag, Berlin.

Einstein, A. (1936). Albert Einstein–Max Born, In *Briefwechsel*, p. 173, (Nymphenburger Verlagshandlung, 1969).

Stellmacher, F. K. (1938). *Mathematische Annalen*, **115**, 740.

Treder, H.-J. (1962). *Gravitative Stosswellen*, Akademie-Verlag, Berlin.

[1] "I have found, together with a young co-worker, that there are no gravitational waves, in spite of the fact that—in conformity with the first approximation—they were considered a certainty." (translator).